Coordinated Science Laboratory

UNIVERSITY OF ILLINOIS – URBANA, ILLINOIS

ANTENNA RESOLUTION AS LIMITED BY ATMOSPHERIC TURBULENCE**

Prepared by
C.M. Angulo[1] & J.P. Ruina

REPORT R-96 JULY 1957

Reissued JANUARY 1960

COORDINATED SCIENCE LABORATORY
UNIVERSITY OF ILLINOIS
URBANA, ILLINOIS

Contract DA-36-039-SC-56695
D/A Sub-Task 3-99-06-111

"The research reported in this document was made possible by support extended to the University of Illinois, Coordinated Science Laboratory, jointly by the Department of the Army (Signal Corps and Ordnance Corps), Department of the Navy (Office of Naval Research), and the Department of the Air Force (Office of Scientific Research, Air Research and Development Command), under Signal Corps Contract DA 36-039-SC-56695.

(1) Division of Engineering, Brown University, Providence, Rhode Island.

**This report was presented orally on May 23, 1957 at the URSI meeting in Washington, D. C.

Numbered Pages: 51

ABSTRACT

Spatial variations of the index of refraction of the atmosphere introduce fluctuations in the phase and amplitude of a wave propagating through it. The effect of these fluctuations on the resolution capability of microwave antennas is discussed in this paper. The measure of resolution used in the calculations is the beamwidth which is defined as the square root of the second moment about the mean of the normalized antenna pattern. This measure simplifies the calculations greatly and results in a simple expression for the beam broadening due to the variation of the refractive index. This broadening is dependent on the distance to the target, the scale of the turbulence, the variance of the refractive index and is relatively insensitive to the shape of the spatial correlation function of the refractive index. Measurements by others of the nature of the refractive index spatial variation are then used to obtain numerical estimates of the ultimate resolution capability of a microwave antenna as limited by atmospheric turbulence.

Table of Contents

Notation

b	-beamwidth
b_o	-unperturbed beamwidth
$C(r)$	-normalized autocorrelation function of index of refraction
$G(\theta)$	-antenna amplitude beam pattern
$\lvert G(\theta)\rvert^2$	-antenna power beam pattern
$g(x)$	-antenna excitation function
L	-antenna length
R	-range antenna to target
r_o	-scale of turbulence or correlation distance of refractive index
s	-separation in distance at the antenna of two rays being considered
x	-length coordinate along antenna
$\beta = \frac{d\Delta}{dx}$	-angle of arrival of ray
Δ	-deviation from the mean of optical path length. $2\pi\Delta$ is therefore phase deviation from mean. $\Delta = \int_o^R \delta \, dr$
δ	$-\dfrac{\text{index of refraction deviation from the mean}}{\text{mean index of refration}}$
$\langle\delta^2\rangle$	$-\dfrac{\text{mean square deviation of the index of refraction}}{\text{square of the mean index of refraction}}$
θ	-angle coordinate measured from the perpendicular to antenna
θ_o	-beam center

All lengths in units of wavelength.

I. Introduction

The problem that concerns us is the determination of the resolution limit of an antenna when the limitation is due to the variations in the refractive index in the propagating medium. Normally the resolution capability of an antenna is directly related to its aperture size and in principle can be increased without limit if the mechanical difficulties are conveniently overlooked. However, there is a limit imposed on the resolution if the propagating medium is not homogeneous, and this probably represents the ultimate resolution capability in most cases. We shall examine this limit assuming a line source antenna in a lossless, isotropic medium with a spatially varying refractive index. We shall assume further that this varying refractive index is due to a homogeneous turbulence with a scale much greater than the radiation wavelength.

The irregularities in the earth's atmosphere and the way they cause twinkling of stars (scintillations in color, position and intensity) has been the subject of extensive study by astronomers. Present theories attribute these scintillations to a comparatively thin turbulent layer in the troposphere at an altitude of approximately 4 kilometers.[1] In radio astronomy, fluctuations in the intensity of radio stars which were at first attributed to changes in emission, were later shown to be caused by the earth's atmosphere.[2]

---

1. S. Chandrasekhar, "A Statistical Basis for the Theory of Stellar Scintillation," Monthly Notices of the Royal Astronomical Society, Vol. 112, No. 5, pp. 475-483, 1952.

2. R. N. Bracewell and J. L. Pawsey, Radio Astronomy, Oxford University Press, 1955.

Unfortunately most of the experimental data available from astronomical observations are not applicable to microwave propagation in the tropospphere since optical seeing is concerned with entirely different wavelengths and fluctuations in the intensity of radio stars are due primarily to turbulence in the ionosphere.

II. Effect of Atmospheric Turbulence

The effect produced by turbulence of the atmosphere on the propagation of electromagnetic waves can be seen semi-quantitatively as follows:

Consider a point source of radiation in a medium with a random space variation of index of refraction. Assume also that the spatial autocorrelation function for the refractive index is a function only of the distance between the two points at which the index is measured and not of the location of the points. If the width of the autocorrelation function for the index is r_o* which we assume is much greater than 1, then we can consider the medium as consisting of "blobs" of linear dimensions $\sim r_o$ with statistically independent values for the refractive index.

A ray emanating from the point source after travelling through one blob will have a mean square deviation from the mean of it's phase front equal to $r_o^2 \langle \delta^2 \rangle$, where $\langle \delta^2 \rangle$ is the mean square fluctuation of of the refractive index. At a distance R from the source the ray will have propagated through R/r_o essentially uncorrelated blobs and the mean square deviation from the mean of the phase front is $Rr_o \langle \delta^2 \rangle$.

---

*All distances in this report are measured in units of the radiation wavelength and are therefore dimensionless.

If we assume that the spatial phase correlation of the radiation at a distance R has the same scale as the spatial correlation of the refractive index, then the deviation from the mean of the phase front at each of two points r_o apart and at a distance R from the source is equal to $Rr_o \langle \delta^2 \rangle$. Assuming further that these deviations are statistically independent and present a linear phase front (see Figure 1) between two points separated by the correlation distance r_o we find that the mean square slope of the phase front $\langle \beta^2 \rangle$ is equal to $\frac{2R \langle \delta^2 \rangle}{r_o}$. The slope of the phase front is the angle of arrival of the radiation for small values of the angle of arrival so that its mean square value is equal to the mean square deflection of the propagation direction.

If instead of a point source we have a source of plane waves, the phases of the received radiation at two points a fixed distance apart would be less closely correlated after travelling through the same distance in the turbulent medium. Therefore, the mean square slope of the phase front would be greater than in the case of a point source. In fact, we show later that the mean square angle of arrival for the two cases differs by a factor of three.

III. Antenna Resolution

For our calculations we shall assume a line source antenna. This is a type frequently used as a model when resolution in only one dimension is being considered. The amplitude beam pattern G (θ) in the far field is approximated (neglecting proportionality factors) by the Fourier Transform of the excitation function[3] g(x) as follows:

$$G(\theta) = \int_{-\infty}^{+\infty} g\,(x)\, e^{-j2\pi x\theta} dx \qquad (1)$$

The integral is taken over the complete aperture. Note that x is measured in units of wavelengths and is therefore dimensionless. This formula is generally valid for $\theta << 1$, the region of interest in highly directive antennas. As a consequence of reciprocity this result applies for both transmitting and receiving patterns.

The resolution capability of a microwave antenna[4] is often measured by the reciprocal of the width of the power pattern $|G(\theta)|^2$ and increases directly with antenna length regardless of the particular definition of beamwidth used.

The most frequently used measure of beamwidth is the angular interval between half power points of the beam pattern. This is a quantity most readily measured experimentally. Other beamwidth measures such as the width of a square pattern with the same maximum gain and total power as the antenna being considered, or the angular interval within which a fixed percentage of the total power lies, can be used. However, the measure b which we shall use is the square root of the

---

3. S. Silver, Microwave Antenna Theory and Design, Chapter 6, McGraw-Hill (1949).

4. For a discussion of azimuthal resolution see J. Freedman, "Resolution in Radar Systems," Proc. I.R.E., vol. 39, p. 813 (July, 1951).

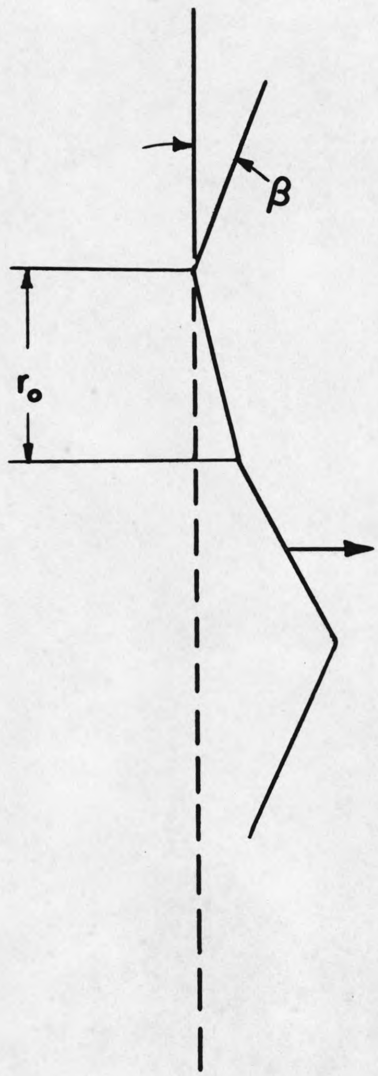

FIG. 1 SIMPLIFIED MODEL OF RADIATION PHASE FRONT

second moment about the mean of the normalized pattern, that is

$$b^2 = \frac{\int_{-\infty}^{+\infty} \theta^2 \left|G(\theta)\right|^2 d\theta}{\int_{-\infty}^{+\infty} \left|G(\theta)\right|^2 d\theta} \tag{2}$$

Using Parseval's theorem and recognizing that $j2\pi\theta G(\theta)$ is the Fourier Transform of the derivative of $g(x)$ we can also write

$$b^2 = \frac{\int_{-\infty}^{+\infty} \left|g'(x)\right|^2 dx}{4\pi^2 \int_{-\infty}^{+\infty} \left|g(x)\right|^2 dx} \tag{3}$$

This measure of width is extremely convenient mathematically and for that reason is almost always used in statistics as a measure of the width of a distribution. The main objection to its use is that it gives undue weight to the size and location of the side lobe power. As an extreme case, an antenna with constant excitation over a finite aperture will have a beamwidth $b = \infty$ regardless of the size of the aperture. In this paper we are primarily concerned with broadening of the beam due to the varying refractive index and as we shall see later there are no infinities in the broadening term even for a uniform aperture illumination. For excitation functions which are not discontinuous in amplitude, b generally agrees well with other beamwidth measures.

Strictly speaking the far or Fraunhofer region beam pattern given by equation (1) holds only at infinite range. The minimum range measured in wavelengths for the applicability of equation (1) is usually considered to be L^2, where L is the antenna length measured in units of wavelength. At this range the phase difference of the contributions of different elements of the antenna is no greater than $\frac{\pi}{4}$. This can be interpreted geometrically by referring to Figure 2A. In the very near region we have the shadow or geometrically beam. In the very far region we have the diffracted beam of width $\sim \frac{1}{L}$. The intersection occurs at a range of L^2. For a 100 ft. antenna operating at $\lambda = 3$ cm, this range is about 20 miles.

To view targets at a range closer than L^2 it is desirable to focus the antenna, that is, to have the radiated phase front be in the form of a circular arc whose center is located at the target of interest. The "depth of field" for this case can be calculated using the same criterion as for the unfocused antenna or from the geometrical construction of Figure 2B.

When $L \ll R$, the beam pattern in the focal plane measured in terms of the subtended angle from the antenna, is the same as the far region pattern of a line source of the same length. Therefore our use of equation (1) will apply to both focused and unfocused antennas.

IV. Beam Broadening Due to Random Phase Variations

In this section we analyze the effect of random phase variations on the beamwidth. The results are independent of the cause of the

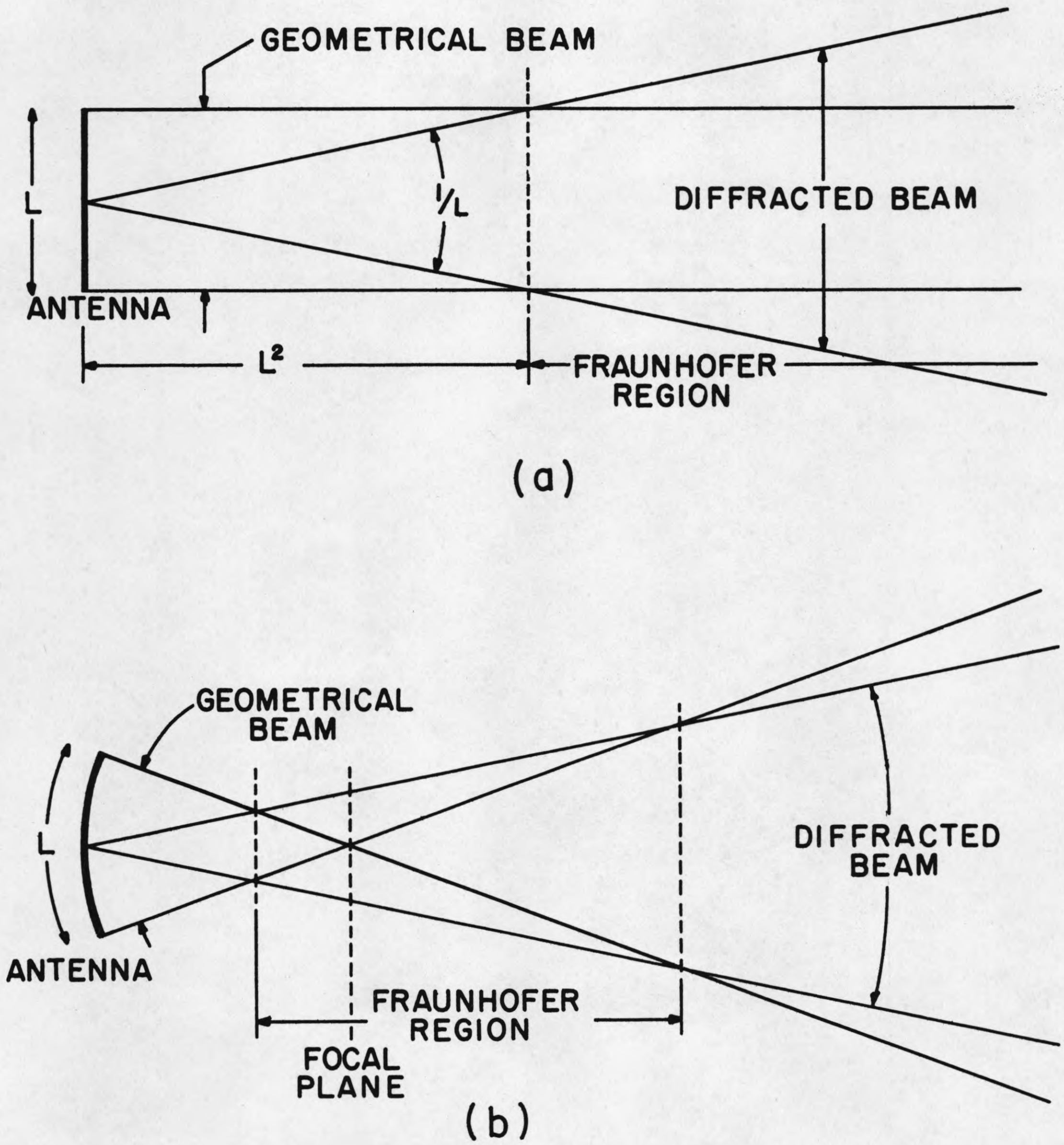

FIG. 2 FOCUSSED AND UNFOCUSSED ANTENNAS

phase variations.

Let us regard the antenna excitation as consisting of the designed excitation $g_o(x)$ which is assumed real, and a perturbing random phase factor $e^{j2\pi\Delta(x)}$. For the problem discussed in this paper $\Delta(x)$ is the deviation from the mean of the path length (measured in wavelengths) from the target to different parts of the antenna. We may write

$$g(x) = g_o(x) \quad e^{j2\pi\Delta(x)} \tag{4}$$

Substituting equation (4) in equation (3) we get

$$b^2 = b_o^2 + \frac{\int_{-\infty}^{+\infty} \beta^2(x)\, g_o^2(x)\, dx}{\int_{-\infty}^{+\infty} g_o^2(x)\, dx} \tag{5}$$

where b_o is the unperturbed beamwidth $[\Delta = 0]$ and $\beta = \frac{d\Delta}{dx}$ can be interpreted as the angle of arrival of the radiation for $\beta \ll 1$.

If we take average values with respect to x of both sides of (5) we obtain

$$\langle b^2 \rangle = b_o^2 + \langle \beta^2 \rangle . \tag{6}$$

The total broadening term $\langle \beta^2 \rangle$ which is simply the mean square angle of arrival may be analyzed into two components. One represents the tilting of the beam pattern and the other the distortion of the beam pattern. To compute these two components, we consider the center of the beam θ_o defined as the first moment of the normalized pattern.

$$\Theta_o = \frac{\int_{-\infty}^{+\infty} \Theta \left| G(\Theta) \right|^2 d\Theta}{\int_{-\infty}^{+\infty} \left| G(\Theta) \right|^2 d\Theta}$$

Since $j2\pi\Theta G(\Theta)$ is the Fourier Transform of $g'(x)$, we can apply Parseval's Theorem and get

$$\Theta_o = \frac{\int_{-\infty}^{+\infty} g^*(x)g'(x)dx}{j2\pi \int_{-\infty}^{+\infty} \left| g(x) \right|^2 dx}$$

Substituting for $g'(x)$ and $g^*(x)$ obtained from equation (4) we have

$$\Theta = \frac{\int_{-\infty}^{+\infty} j2\pi\Delta'(x)g_o^2(x)dx + \int_{-\infty}^{+\infty} g_o(x)g_o'(x)dx}{j2\pi \int_{-\infty}^{+\infty} g_o^2(x)dx}$$

But the second term in the numerator is zero so that we are left with

$$\Theta_o = \frac{\int_{-\infty}^{+\infty} \beta(x)g_o^2(x)dx}{\int_{-\infty}^{+\infty} g_o^2(x)dx}$$

From this we get $<\theta_o> = <\beta> = 0$ as expected, and

$$<\theta_o^2> = \frac{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} <\beta(x)\,\beta(y)>\, g_o^2(x)\, g_o^2(y)\, dxdy}{\left[\int_{-\infty}^{+\infty} g_o^2(x)dx\right]^2} \tag{7}$$

The value of $<\theta_o^2>$ varies from zero when the spatial correlation length for β is zero to $<\beta^2>$ when the correlation length associated with β is much greater than the antenna length.

The second moment of the beam pattern about its center is $b^2 - \theta_o^2$ and its mean value $<b^2> - <\theta_o^2>$ is equal to $b_o^2 + <\beta^2> - <\theta_o^2>$ as we can see from equation (6). From this we see that the broadening due to distortion of the beam is $<\beta^2> - <\theta_o^2>$. The broadening due to tilt is of course $<\theta_o^2>$ and the sum of the two terms is $<\beta^2>$, the total broadening.

When the correlation length of β is large compared to the antenna length the broadening $<\beta^2>$ is due almost completely to the tilt contribution while for a very small correlation length for β the main contribution is due to beam distortion.

V. Results and Conclusions

We recognize that the knowledge of $<\beta(x)\,\beta(x+s)>$ is absolutely essential to the calculation of the tilt. Table I has the expressions for $<\triangle(x)\triangle(x+s)>$ and $<\beta(x)\,\beta(x+s)>$ corresponding to three different types of $C(r_{12})$. The results for two of the three cases have been already given by others [1,5] and it has been found that $\langle \beta(x)\, \beta\,(x+s)\rangle$ is not very sensitive to the type of correlation function assumed for the dielectric constant. We have only computed in this Report one typical curve to show this property (see Figure 3).

No further discussion of the phase correlation and angle of arrival correlation seem to be necessary here in view of the work already published on the subject [1,5]. However, our derivation of $\langle \Delta(x)\, \Delta(x+s)\rangle$ and $\langle \beta(x)\, \beta\,(x+s)\rangle$ is so simple that we consider of interest to include it in Appendix A.

The expression (7) which is as a measure of the tilting of the beam and its complement to one which is a measure of the broadening around the center of the beam are, of course, functions of the illumination. We have calculated the exact expressions for Gaussian illumination, for two different autocorrelation functions of the dielectric constant. The mathematical details are given in Appendix B and we will give here only the results.

For $g_o(x) = e^{-2\frac{x^2}{L^2}}$ and $C(r) = e^{-\frac{r^2}{r_o^2}}$ we obtain

$$\frac{\langle \theta_o^{\,2} \rangle}{\langle \beta^2 \rangle} = \left[\frac{L^2}{2r_o^{\,2}} + 1 \right]^{-\frac{3}{2}} \qquad (8)$$

for parallel rays, and

$$\frac{\langle \theta_o^{\,2} \rangle}{\langle \beta^2 \rangle} = \frac{r_o^{\,3}}{L^3}\, 6\sqrt{2} \left\{ \ln. \frac{L}{r_o\sqrt{2}} + \left(\frac{L^2}{2r_o^{\,2}} + 1 \right)^{\frac{1}{2}} - \frac{L}{r_o\sqrt{2}} \left(\frac{L^2}{2r_o^{\,2}} + 1 \right)^{-\frac{1}{2}} \right\} \qquad (9)$$

---

5. R. B. Muchmore and A. D. Wheelon, "Line of sight propagation phenomena" Proc. I.R.E., Vol. 43, pp. 1437 and 1450; October, 1959.

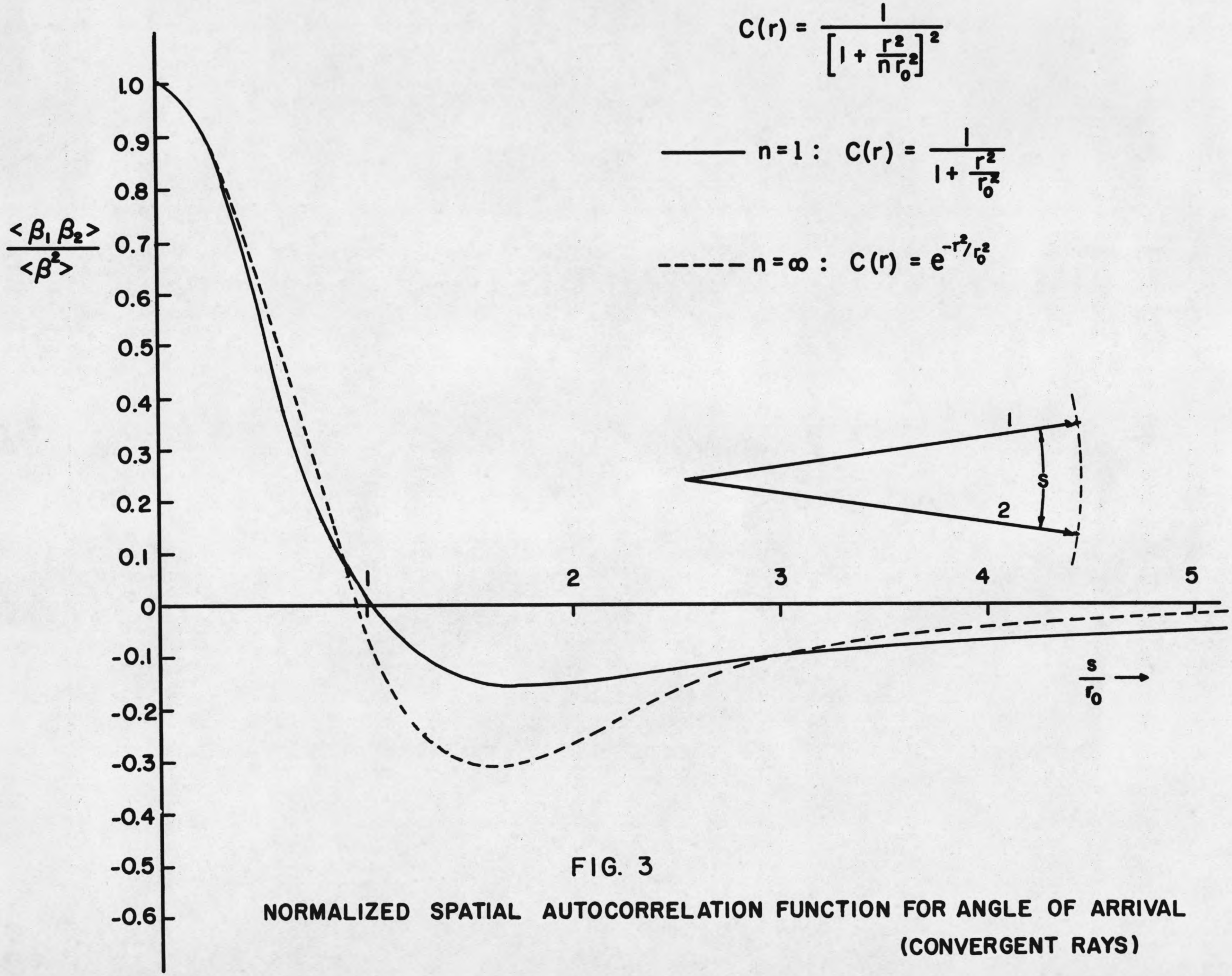

FIG. 3

NORMALIZED SPATIAL AUTOCORRELATION FUNCTION FOR ANGLE OF ARRIVAL (CONVERGENT RAYS) for convergent rays.

For $g_o(x) = e^{-2\frac{x^2}{L^2}}$ and $C(r) = \dfrac{1}{1 + \dfrac{r^2}{r_o^2}}$ we obtain

$$\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle} = \frac{r_o^3}{L^3} e^{\frac{r_o^2}{L^2}} 8\sqrt{\frac{2}{\pi}} \left\{ \left(\frac{r_o^2}{L^2} + \frac{1}{2} \right) K_o^2 \left(\frac{r_o^2}{L^2} \right) - \frac{r_o^2}{L^2} K_1^2 \left(\frac{r_o^2}{L^2} \right) \right\} \quad (10)$$

for parallel rays, and

$$\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle} = \frac{r_o^3}{L^3} 6\sqrt{\frac{2}{\pi}} \left\{ e^{\frac{r_o^2}{L^2}} \left[K_1\left(\frac{r_o^2}{L^2} \right) - K_o\left(\frac{r_o^2}{L^2} \right) \right] - \int_{\frac{r_o^2}{L^2}}^{\infty} \frac{e^x K_1(x) - e^x K_o(x)}{x} dx \right\} \quad (11)$$

for convergent rays.

In the above expressions K_o and K_1 represent the modified Bessel's Functions of the second kind of zero and first order respectively.

A plot of expressions (8) and (9) is given in Figure 4.

If the antenna is smaller than the correlation length of the dielectric constant fluctuation we can obtain simple expressions for $\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle}$. These expressions separate into a factor depending only on the correlation function of the dielectric constant fluctuation and a factor which depends only on the illumination. The mathematics of the approximations are carried out in Appendix A. The dominant terms of the expressions are the following:

$$\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle} = 1 + 3 \frac{\int_{-\infty}^{\infty} \frac{1}{r} \frac{d}{dr} \left[\frac{C'(r)}{r} \right] dr}{\int_{-\infty}^{\infty} \frac{C'(r)}{r} dr} \quad \frac{\int_{-\infty}^{\infty} x^2 g_o^2(x) dx}{\int_{-\infty}^{\infty} g_o^2(x)\, dx} \qquad (12)$$

for parallel rays, and

$$\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle} = 1 + \frac{9}{5} \frac{\int_{-\infty}^{\infty} \frac{1}{r} \frac{d}{dr} \left[\frac{C'(r)}{r} \right] dr}{\int_{-\infty}^{\infty} \frac{C'(r)}{r} dx} \quad \frac{\int_{-\infty}^{\infty} x^2 g_o^2(x) dx}{\int_{-\infty}^{\infty} g_o^2(x) dx} \qquad (13)$$

for convergent rays.

The approximate values given above have been compared with the exact expressions for the case of Gaussian correlation and Gaussian illumination. They agree to the second significant figure for a scale of turbulence $r_o \geqslant 3.5L$.

As the plot of Figure 4 indicates the results for convergent and parallel rays are practically the same for any scale of turbulence larger than 1.8L. Moreover we can also see in the plot that the tilting term is the dominant one if the length of the antenna remains smaller than half the scale of the turbulence. Among the experimental values found for the scale of turbulence 200 feet is not a large value. In this case, an antenna of 100 feet length would only experience beam broadening due to tilting.

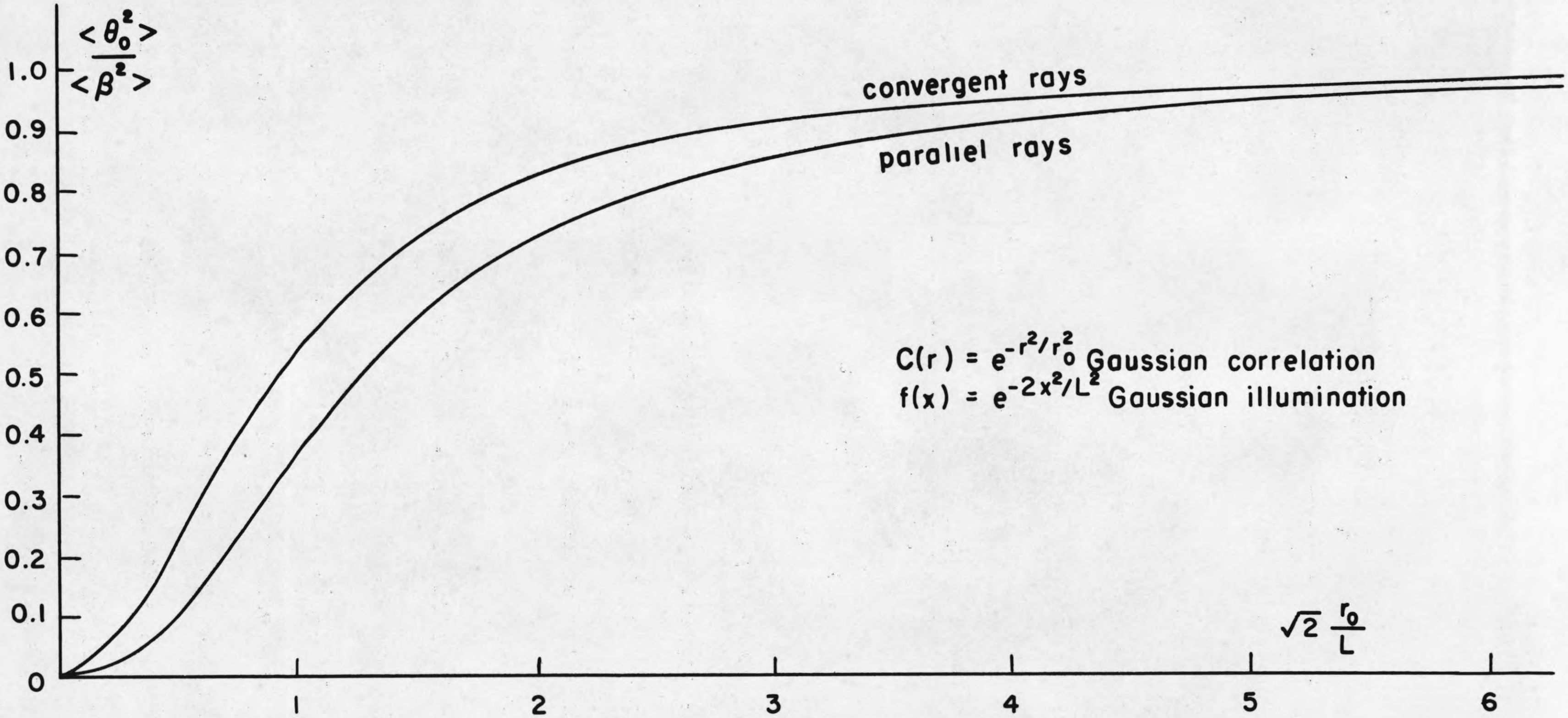

FIG. 4 THE TILT

96-27

The usefulness of the results depend on the experimental knowledge of the scale of the turbulence. The authors are not aware of any comprehensive measurements which have given conclusive results on the validity of the physical model of the troposphere assumed in this paper. With the present general interest in propagation problems however we have no doubt that our knowledge will be greatly increased in the near future.

The principal reason of this report is to find a criterion for the limitations in resolution due to fluctuations of the index of refraction. A possible criterion is that the antenna length must not exceed the value which makes equal the two terms of the right hand side of equation (6). In other words: The length of the antenna should be less than the value for which the total broadening effect $\langle \beta^2 \rangle$ is equal to the square of the beamwidth in the absence of the fluctuations of the dielectric constant. For an antenna with Gaussian illumination $g_o(x) = e^{-2\frac{x^2}{L^2}}$ equation (3) gives:

$$b_o^2 = \frac{1}{2\pi^2 L^2}$$

For a Gaussian spatial Auto-correlation function of Refractive index and a focused antenna (convergent rays) Table 1 gives:

$$\langle \beta^2 \rangle = \frac{2\sqrt{\pi}}{3} \langle \delta^2 \rangle \frac{R}{r_o}$$

The two expressions are equal if

$$L^2 = \frac{0.043}{\frac{\langle \delta^2 \rangle}{r_o} R}$$

If we take for[6] $\frac{\langle \delta^2 \rangle}{r_o} = 7 \times 10^{-16}$/feet we obtain:

$$L = \frac{107500}{R^{1/2}} \text{ miles}$$

which gives for R = 100 miles:

$$L = 10750\ \lambda \text{ or } L \approx 322 \text{ meters in the x band}$$

If the antenna is not focused Table I gives

$$\langle \beta^2 \rangle = 2\sqrt{\pi}\ \langle \delta^2 \rangle\ \frac{R}{r_o}$$

and therefore we obtain

$$L = \frac{62000}{R^{1/2}} \text{ miles}$$

which gives for R = 100 miles:

$$L = 6200\ \lambda \text{ or } L \approx 200 \text{ meters in the x band.}$$

For a 200 or 300 meters antenna at 3 cm the minimum range for the Fraunhofer region is

$$R = L\frac{L}{\lambda} > 800 \text{ miles}$$

Therefore we should use the results of the focused antenna. That is, to look at a distance of 100 miles the tropospheric turbulence allows a focused antenna 300 meters long.

---

6. C. M. Crain, A. W. Straiton and C. E. von Rosenberg, "A Statistical Survey of Atmospheric Index of Refraction Variation". I.R.E. Transactions on Antennas and Propagation, Vol. AP-1 pp. 43-43; October, 1953.

Appendix A

This Appendix contains the derivation of general mathematical expressions. No specific functions are assumed for the correlation function of the index of refraction fluctuation $C(r_{12})$ nor for the antenna illumination. The derivations included are the following:

A very simple derivation of $\langle \Delta(x)\,\Delta(x+s)\rangle$ and $\langle \beta(x)\beta(x+s)\rangle$ in terms of a simple integral involving $C(r_{12})$.

Approximations to $\langle \Delta(x)\,\Delta(x+s)\rangle$, $\langle \beta(x)\beta(x+s)\rangle$ and $\langle \theta_o^2\rangle$ for small values of s (small antenna compared with the scale of the turbulence). This approximate expressions exhibit independently the effects of $C(r_{12})$ and the illumination on the tilting.

A. 1. Correlation function of $\langle \Delta(x)\,\Delta(x+s)\rangle$.

Let us consider the two convergent rays represented in Figure 5. The correlation function of the optical path fluctuation is obviously given by

$$\frac{\langle \Delta(x)\Delta(x+s)\rangle}{\langle \delta^2\rangle\; r_o R} = \frac{\left\langle \int_o^R \delta(r_2)\,dr_2 \int_o^R \delta(r_1)\,dr_1 \right\rangle}{\langle \delta^2\rangle\; r_o R} =$$

$$\frac{\int_o^R dr_2 \int_o^R \langle \delta(r_1)\;\delta(r_2)\rangle\, dr_1}{\langle \delta^2\rangle\; r_o R} \tag{14}$$

Where $\delta(r)$ represents the fluctuation of the index of refraction in the turbulent medium.

$$n = 1 + \delta(r)$$

The constant denominator chosen as a normalization factor simplifies the mathematical results for the expression of the correlation function. The physical significance of the normalization factor has been shown in Part II of this Report.

The correlation function of the index of refraction fluctuation is defined as:

$$C(r_{12}) = \frac{\langle \delta(r_1)\,\delta(r_2) \rangle}{\langle \delta^2 \rangle} \tag{15}$$

Substituting (15) into (14) we obtain

$$\frac{\langle \Delta(x)\,\Delta(x+s) \rangle}{\langle \delta^2 \rangle\, r_o R} = \frac{\int_o^R dr_1 \int_o^R C(r_{12})\, dr_2}{r_o R} \tag{16}$$

It is easy to see from Figure 5 that, for small angle ϕ_1

$$r_{12} = \sqrt{(r_2 - r_1)^2 + \left(\frac{r_1 s}{R}\right)^2}$$

The range R is much larger than the correlation length of the index of refraction fluctuation, therefore we can extend the limits of integration of the first integral from $-\infty$ to $+\infty$. Expression (16) becomes

$$\frac{\langle \Delta(x)\,\Delta(x+s) \rangle}{\langle \delta^2 \rangle\, r_o R} = \frac{\int_o^R dr_1 \int_{-\infty}^{\infty} C\left[\sqrt{(r_1 - r_2)^2 + \left(\frac{r_1 s}{R}\right)^2}\right] dr_2}{r_o R} \tag{17}$$

If we call $r_2 - r_1 = r$ the expression (17) takes finally the form

$$\frac{\langle \Delta(x)\,\Delta(x+s) \rangle}{\langle \delta^2 \rangle r_o R} = \frac{\int_o^R dr_1 \int_{-\infty}^{\infty} C\left[\sqrt{r^2 + \left(\frac{r_1 s}{R}\right)^2}\right] dr}{r_o R} \tag{18}$$

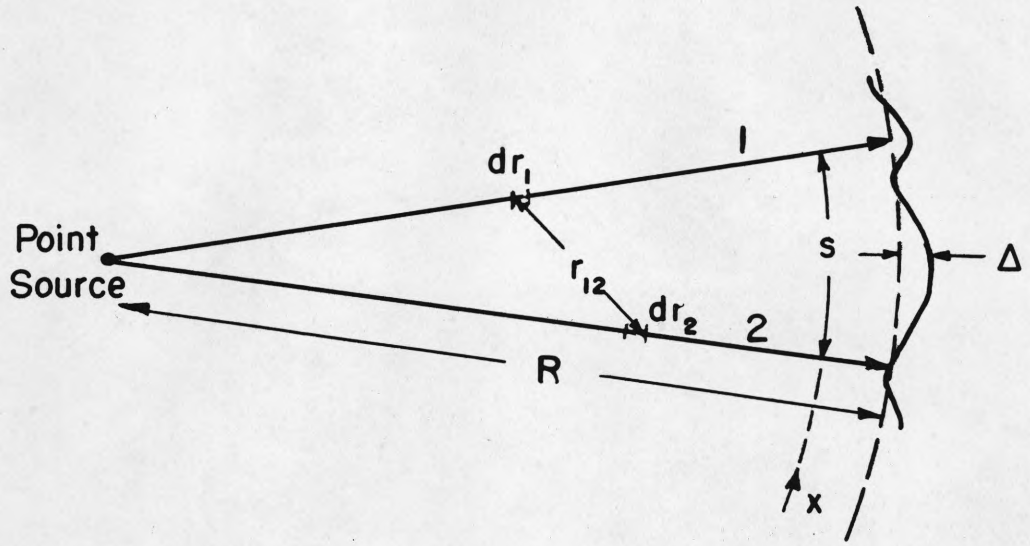

FIG. 5

The equation (18) is the general expression given in Table I for the correlation of the optical path fluctuation for convergent rays. It can be used for any correlation function of the index of refraction fluctuation.

If we consider two parallel rays, the derivation of a general expression for $\langle \Delta(x)\Delta(x+s)\rangle$ is still easier. We proceed as in the case of convergent rays but now

$$r_{12} = \sqrt{\left(r_2 - r_1\right)^2 + s^2}$$

Therefore, instead of the equation (17) we obtain the following one

$$\frac{\langle \Delta(x)\,\Delta(x+s)\rangle}{\langle \delta^2\rangle r_o R} = \frac{\int_o^R dr_1 \int_{-\infty}^{\infty} C\left[\sqrt{r^2+s^2}\right] dr}{r_o R} \qquad (19)$$

The double integral is only a product of simple integrals and we obtain finally

$$\frac{\langle \Delta(x)\,\Delta(x+s)\rangle}{\langle \delta^2\rangle r_o R} = \frac{1}{r_o}\int_{-\infty}^{\infty} C\left[\sqrt{r^2+s^2}\right] dr \qquad (20)$$

The results for the integrals (18) and (20) for three types of correlation functions are tabulated in Table I. The integrations do not present great difficulties for any of the three correlation functions considered[7].

---

7. By simple manipulations it is possible to write the integrals in forms which can be found in "Integraltafel, Zweiter Teil, Bestimmte Integrale" by W. Gröbner and N. Hofreiter, Springer-Verlog.

The results for Gaussian correlation have been obtained by direct integration and also by making $n \longrightarrow \infty$.

A. 2. Correlation function of $<\beta(x)\ \beta(x+s)>$.

The angle of arrival fluctuation is the derivative of the optical path length with respect to the coordinate along the antenna

$$\beta(x) = \frac{d\Delta}{dx}$$

Therefore the correlation function of the angle of arrival fluctuation is given by

$$<\beta(x_1)\beta(x_2)> = <\left(\frac{d}{dx_1}\int_o^R \delta(r_1)\,dr_1\right)\left(\frac{d}{dx_2}\int_o^R \delta(r_2)\,dr_2\right)> \tag{21}$$

With the notation of Figure 5.

$$s = x_1 = x_2$$

$$\frac{d}{dx_1} = \frac{d}{ds}\ ;\ \frac{d}{dx_2} = -\frac{d}{ds}$$

Equation (21) can be now written as

$$<\beta(x)\beta(x+s)> = \frac{d^2}{ds^2} < \int_o^R \delta(r_1)\,dr_1 \int_o^R \delta(r_2)\,dr_2 >$$

Therefore:

$$\frac{<\beta(x)\beta(x+s)>}{<\delta^2>\frac{R}{r_o}} = -r_o^{\,2}\ \frac{d^2}{ds^2}\ \frac{<\Delta(x)\ \Delta(x+s)>}{<\delta^2>r_oR}$$

Finally we obtain the general expressions given in Table I.

$$\frac{<\beta(x)\beta(x+s)>}{<\delta^2>\frac{R}{r_o}} = -r_o \quad \frac{d^2}{ds^2} \int_{-\infty}^{+\infty} C\left[\sqrt{r^2 + s^2}\right] dr \tag{22}$$

for parallel rays, and

$$\frac{<\beta(x)\beta(x+s)>}{<\delta^2>\frac{R}{r_o}} = -\frac{r_o}{R} \quad \frac{d^2}{ds^2} \int_{o}^{R} dr_1 \int_{-\infty}^{\infty} C\left[\sqrt{r^2 + \frac{r_1^2 \ s^2}{R^2}}\right] dr \tag{23}$$

for convergent rays.

The correlation functions for the angle of arrival fluctuation do not require an integration. Only a simple differentiation of the optical path fluctuation is necessary. The results for three different types of correlation function of the index of refraction fluctuation are tabulated in Table I.

A. 3. Approximations to the general expressions of $<\Delta(x)\,\Delta(x+s)>$ $<\beta(x)\beta(x+s)>$ and $<\theta_o^2>$ for small antennas:

We derive here an approximation to equations (18), (20), (22), (23) and (7) for the case of s small.

The approximations to (18), (20), (22) and (23) are quite simple and we will not give the details. It is assumed that the correlation function of the index of refraction fluctuation and all its derivatives are well defined for any real value of r. It is further assumed that the integrals which appear in the derivation exist. Under those assumptions we expand in a MacLaurin series around s = o and obtain for parallel rays the following expressions:

$$\frac{<\Delta(x)\,\Delta(x+s)>}{<\Delta^2>} = 1 + \frac{A_1}{2}s^2 + \frac{A_2}{8}s^4 + \text{-----}$$

$$\frac{<\beta(x)\beta(x+s)>}{<\beta^2>} = 1 + \frac{3}{2}\ A_3\ s^2 + \text{-----} \tag{24}$$

The constants are the following

$$A_1 = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{1}{r}\ \frac{dC(r)}{dr}\ dr}{\displaystyle\int_{-\infty}^{\infty} C(r)\ dr}$$

$$A_2 = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{1}{r}\ \frac{d}{dr}\left[\frac{1}{r}\ \frac{dC(r)}{dr}\right] dr}{\displaystyle\int_{-\infty}^{\infty} C(r)\ dr}$$

$$A_3 = \frac{\displaystyle\int_{-\infty}^{-\infty} \frac{1}{r}\ \frac{d}{dr}\left[\frac{1}{r}\ \frac{dC(r)}{dr}\right] dr}{\displaystyle\int_{-\infty}^{\infty} \frac{1}{r}\ \frac{dC(r)}{dr}\ dr}$$

For convergent rays the approximations are:

$$\frac{<\Delta(x)\Delta(x+s)>}{<\Delta^2>} = 1 + \frac{A_1}{6} s^2 + \frac{A_2}{40} s^4 + ----$$

$$\frac{<\beta(x)\beta(x+s)>}{<\beta^2>} = 1 + \frac{9}{10} A_3^{\;2} + ---- \tag{25}$$

If we insert the approximations given in (24) and (25) into equation (7) we obtain:

$$\frac{<\theta_o^{\;2}>}{<\beta^2>} = 1 + 3 A_3 \gamma \text{ for parallel rays} \tag{26}$$

$$\frac{<\theta_o^{\;2}>}{<\beta^2>} = 1 + \frac{9}{5} A_3\gamma \text{ for convergent rays} \tag{27}$$

The constant γ is a function of the illumination

$$\gamma = \frac{1}{2} \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x-y)^2 g_o^{\;2}(x)\, g_o^{\;2}(y)\, dxdy}{\left[\int_{-\infty}^{\infty} g_o^{\;2}(x)\, dx\right]^2}$$

By expanding γ we obtain

$$\gamma = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^2 g_o^{\;2}(x)\, g_o^{\;2}(y)\, dxdy}{\left[\int_{-\infty}^{\infty} g_o^{\;2}(x)\, dx\right]^2} - \frac{\left[\int_{-\infty}^{\infty} xg_o^{\;2}(x)\, dx\right]^2}{\left[\int_{-\infty}^{\infty} g_o^{\;2}(x)\, dx\right]^2} \tag{28}$$

If we impose on the square of the illumination the only condition of being even the second term of (28) is zero, and therefore

$$\gamma = \frac{\int_0^\infty x^2 \, g_o^{\,2} (x) \, dx}{\int_0^\infty g_o^{\,2} (x) \, dx}$$

The expressions (26) and (27) exhibit separately the dependence of the correlation of the index of refraction fluctuation and the dependence on the illumination.

Appendix B

In this Appendix we have included the calculation of the general expressions given in Appendix A for $\frac{<\theta_o^2>}{<\beta^2>}$ in the particular case of Gaussian illumination of the antenna and two different types of Correlation function of the index of refraction fluctuation.

B.1 $\frac{<\theta_o^2>}{<\beta^2>}$ for Parallel Rays, Gaussian $g_o(x)$ and Gaussian $C(r_{12})$.

We obtain first the expression for $\frac{<\beta(x)\ (y)>}{<\beta^2>}$ from Table I. Next we insert it in equation (7) and we also insert in (7) the expression $g_o(x) = e^{-2\frac{x^2}{L^2}}$. We obtain the following expression:

$$\frac{<\theta_o^2>}{<\beta^2>} = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-4\frac{x^2+y^2}{L^2}} \left[1-2\left(\frac{x-y}{r_o}\right)^2\right] e^{-\left(\frac{x-y}{r_o}\right)^2} dxdy}{\frac{\pi}{4}\ L^2}$$

The following change of variables

$$\begin{aligned} \sqrt{2}\, x &= v + w \\ \sqrt{2}\, y &= v - w \end{aligned} \tag{29}$$

transforms the double integral into a product of two simple integrals which can be easily evaluated. The final result is given in Section V of the Report.

B.2. $\frac{<\theta_o^2>}{<\beta^2>}$ for Convergent Rays, Gaussian $g_o(x)$ and Gaussian $C(r_{12})$.

After the insertion of the integrands the expression to be evaluated becomes

$$\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle} = \frac{4}{\pi L^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{\left\{ -4 \frac{x^2+y^2}{L^2} - \left(\frac{x-y}{r_o}\right)^2 \right\}} \left[1 + \left(\frac{r_o}{x-y}\right)^2 \right. - \left. \frac{\int_o^{\frac{x-y}{r_o}} e^{-a^2} \, d\alpha}{\left(\frac{x-y}{r_o}\right)^2} \right] dxdy \tag{30}$$

This expression can be rewritten in a more convenient form if we make use of the following integration by parts

$$e^{-a^2} \left(1 + \frac{1}{a^2}\right) - \frac{1}{a^3} \int_o^a e^{-\alpha^2} d\alpha = e^{-a^2} - 2 \frac{\int_o^a \alpha^2 e^{-\alpha^2} d\alpha}{a^3}$$

The expression (30) becomes the following one

$$\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle} = \frac{4}{L^2} \int\int_{-\infty}^{\infty} e^{-4\frac{x^2+y^2}{L^2}} e^{-\left(\frac{x-y}{r_o}\right)^2} dxdy - \frac{8}{\pi L^2} \int\int_{-\infty}^{\infty} e^{-4\frac{x^2+y^2}{L^2}} \frac{\int_o^{\frac{x-y}{r_o}} \alpha^2 e^{-\alpha^2} d\alpha}{\left(\frac{x-y}{r_o}\right)^3} dxdy \tag{31}$$

Of the two double infinite integrals of the expression (31) the first one is very easy to integrate by using the change of variables given in equations (29). We will leave up to the reader the details of this simple integration.

The second double integral is somewhat more complicated. By the change of variables given in equations (29) it transforms as follows

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-4\frac{x^2+y^2}{L^2}} \frac{\int_0^{\frac{x-y}{r_o}} \alpha^2 e^{-\alpha^2} d\alpha}{(\frac{x-y}{r_o})^3} dxdy = \tag{32}$$

$$= \int_{-\infty}^{\infty} e^{-4\frac{v^2}{L^2}} dv \int_{-\infty}^{\infty} e^{-4\frac{w^2}{L^2}} \frac{\int_0^{\frac{\sqrt{2}}{r_o} w} \alpha^2 e^{-\alpha^2} d\alpha}{\frac{2\sqrt{2}}{r_o^3} w^3} dw$$

The integration of the first factor in the right hand member of equation (32) is obvious. The second factor can be written in the following form

$$\int_{-\infty}^{\infty} e^{-4\frac{w^2}{L^2}} \frac{\int_0^{\frac{\sqrt{2}}{r_o} w} \alpha^2 e^{-\alpha^2} d\alpha}{\frac{2\sqrt{2}}{r_o^3} w^3} dw = \frac{r_o^3}{2\sqrt{2}} F(\frac{\sqrt{2}}{r_o})$$

where

$$F(z) = \int_{-\infty}^{\infty} e^{-\frac{4w^2}{L^2}} \frac{\int_0^{zw} \alpha^2 e^{-\alpha^2} d\alpha}{w^3} dw$$

The derivative of F(z) can be integrated easily.

$$\frac{dF(z)}{dz} = z^2 \int_{-\infty}^{\infty} e^{-(\frac{4}{L^2} + z^2) w^2} dw = \frac{\sqrt{\pi}\, z^2}{\sqrt{z^2 + \frac{4}{L^2}}}$$

The value of F(o) is obviously zero. Therefore:

$$F(z) = \int_0^z \sqrt{\pi}\,\frac{z^2}{\sqrt{z^2 + \frac{4}{L^2}}}\,dz = \sqrt{\pi}\left\{\frac{z\sqrt{z^2 + \frac{4}{L^2}}}{2} - \frac{4}{2L^2}\,\ell n\left(\frac{z + \sqrt{\frac{4}{L^2} + z^2}}{\frac{2}{L}}\right)\right\}$$

The final result for the integration of expression (32) is therefore:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-4\frac{x^2+y^2}{L^2}}\;\frac{\int_0^{\frac{x-y}{r_o}} \alpha^2 e^{-\alpha^2}\,d\alpha}{\left(\frac{x-y}{r_o}\right)^3}\,dxdy =$$

$$= \frac{\pi r_o^2}{4}\left\{\sqrt{1 + \frac{L^2}{2r_o^2}} - \frac{\left(\ell n\,\frac{L}{\sqrt{2}r_o} + \sqrt{1 + \frac{L^2}{2r_o^2}}\right)}{\frac{L}{\sqrt{2}\,r_o}}\right\}$$

The value of the first double integral in expression (31) is

$$\iint_{-\infty}^{\infty} e^{-4\frac{x^2+y^2}{L^2}}\; e^{-\left(\frac{x-y}{r_o}\right)^2}\,dxdy = \frac{\pi L^2}{4}\;\frac{1}{\sqrt{1+\frac{L^2}{2r_o^2}}}$$

It is now trivial to combine the partial results derived above and obtain the final answer given in Section V of the Report.

B.3. $\frac{\langle \theta_o^2\rangle}{\langle \beta^2\rangle}$ for Parallel Rays, Gaussian $g_o(x)$ and $C(r_{12})$ of the Cauchy's type.

Substituting the appropriate expressions for the integrands, equation (7) becomes:

$$\frac{\langle \theta_o^2\rangle}{\langle \beta^2\rangle} = \frac{4}{\pi L^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-4\frac{x^2+y^2}{L^2}}\;\frac{1-2\left(\frac{x-y}{r_o}\right)^2}{\left[1 + \left(\frac{x-y}{r_o}\right)^2\right]^{\frac{5}{2}}}\,dxdy$$

By the change of variables given in equations (29) we obtain

$$\frac{\langle\theta_o^2\rangle}{\langle\beta^2\rangle} = \frac{4}{\pi L^2}\int_{-\infty}^{\infty} e^{-\frac{4}{L^2}v^2}\, dv\, \frac{r_o^3}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-\frac{4}{L^2}w^2}\, \frac{\frac{r_o^2}{4} - w^2}{\left(\frac{r_o^2}{2} + w^2\right)^{\frac{5}{2}}}\, dw$$

The first integral is equal to $\sqrt{\pi}\,\frac{L}{2}$.

The second integral can be written in a more convenient form after simple algebraic transformations.

$$\int_{-\infty}^{\infty} e^{-\frac{4}{L^2}w^2}\, \frac{\frac{r_o^2}{4} - w^2}{\left(\frac{r_o^2}{2} + w^2\right)^{\frac{5}{2}}}\, dw = \frac{3}{4}\, r_o^2 \int_{-\infty}^{\infty} \frac{e^{-\frac{4}{L^2}x^2}}{\left[\frac{r_o^2}{2} + x^2\right]^{\frac{5}{2}}}\, dx - \int_{-\infty}^{\infty} \frac{e^{-\frac{4}{L^2}x^2}\, dx}{\left[\frac{r_o^2}{2} + x^2\right]^{\frac{3}{2}}} \qquad (33)$$

But,* we know

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{4}{L^2}x^2}}{\left[\frac{r_o^2}{2} + x^2\right]^{\frac{1}{2}}}\, dx = e^{\frac{r_o^2}{L^2}}\, K_o\left(\frac{r_o^2}{L^2}\right)$$

By successive differentiations respect to r_o we can obtain the two integrals in the right hand member of (33)

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{4}{L^2}x^2}\, dx}{\left(\frac{r_o^2}{2} + x^2\right)^{\frac{3}{2}}} = -\frac{4}{L^2}\, e^{\frac{r_o^2}{L^2}} \left\{K_o\left(\frac{r_o^2}{L^2}\right) - K_1\left(\frac{r_o^2}{L^2}\right)\right\}$$

---

*See Reference 7, p. 65.

and

$$\int_{-\infty}^{\infty} \frac{e^{-\frac{4}{L^2}x^2}\, dx}{\left(\frac{r_o^2}{2} + x^2\right)^{\frac{5}{2}}} = \frac{16}{3L^4} \frac{d}{d\left(\frac{r_o^2}{L^2}\right)} \left\{ e^{\frac{r_o^2}{L^2}} K_o\left(\frac{r_o^2}{L^2}\right) - e^{\frac{r_o^2}{L^2}} K_1\left(\frac{r_o^2}{L^2}\right) \right\}$$

In the above expressions $K_o(x)$ and $K_1(x)$ are the modified Bessel Functions of the second kind, of zero and first order respectively.

The final result for the integration of equation (33) is, therefore

$$\int_{-\infty}^{\infty} e^{-\frac{4}{L^2} w^2} \frac{\frac{r_o^2}{4} - w^2}{\left(\frac{r_o^2}{2} + w^2\right)^{\frac{5}{2}}} dw = \frac{4}{L^2} e^{\frac{r_o^2}{L^2}} \left\{ \left(\frac{2r_o^2}{L^2} + 1\right) K_o\left(\frac{r_o^2}{L^2}\right) - 2\frac{r_o^2}{L^2} K_1\left(\frac{r_o^2}{L^2}\right) \right\}$$

It is very easy to combine the results obtained above to write the final expression given in Section 5.

B.4. $\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle}$ for Convergent Rays, Gaussian $g_o(x)$ and $C(r_{12})$ of the Cauchy's type.

As in the previous cases we insert into the equation (7) the expression for $\frac{\langle \beta(x)\beta(y) \rangle}{\langle \beta^2 \rangle}$ obtained from Table I and the Gaussian expression for $g_o(x)$. In the resulting integrand we effect the change of variable indicate in equations (29). The integrals to be calculated are now written as follows:

$$\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle} = \frac{12}{\pi L^2} \int_{-\infty}^{\infty} e^{-\frac{4}{L^2} v^2} dv \int_{-\infty}^{\infty} \frac{e^{-\frac{4}{L^2} w^2}}{\left(1 + 2\frac{w^2}{r_o^2}\right)^{\frac{3}{2}}} dw +$$

$$+\frac{8}{\pi L^2}\int_{-\infty}^{\infty} e^{-\frac{4}{L^2}v^2}\,dv\int_{-\infty}^{\infty} e^{-\frac{4}{L^2}w^2}\,\frac{\dfrac{1}{\left(1+2\dfrac{w^2}{r_o^2}\right)^{\frac{3}{2}}}-\dfrac{\arcsin\dfrac{w\sqrt{2}}{r_o}}{\dfrac{w\sqrt{2}}{r_o}}}{2\dfrac{w^2}{r_o^2}}\,dw$$

The integration in v is trivial.

$$\int_{-\infty}^{\infty} e^{-\frac{4}{L^2}v^2}\,dv = \sqrt{\pi}\,\frac{L}{2}$$

The remaining integral is

$$\frac{\langle\theta_o^2\rangle}{\langle\beta^2\rangle} = 3\sqrt{\frac{2}{\pi}}\;\frac{r_o}{L}\int_{-\infty}^{\infty}\frac{e^{-2\frac{r_o^2}{L^2}x^2}}{(1+x^2)^{\frac{3}{2}}}\,dx\;+$$

$$+\,2\sqrt{\frac{2}{\pi}}\;\frac{r_o}{L}\int_{-\infty}^{\infty} e^{-2\frac{r_o^2}{L^2}x^2}\,\frac{\dfrac{1}{(1+x^2)^{\frac{3}{2}}}-\dfrac{\operatorname{arc\,sinh} x}{x}}{x^2}\,dx \tag{34}$$

The integrand in the second term can be written in a better way

$$\frac{\dfrac{1}{(1+x^2)^{\frac{3}{2}}}-\dfrac{\operatorname{arc\,sinh} x}{x}}{x^2} = \frac{\dfrac{x}{(1+x^2)^{\frac{3}{2}}}-\displaystyle\int_o^x\frac{dy}{\sqrt{1+y^2}}}{x^3} =$$

$$= \frac{-\dfrac{x^3}{(1+x^2)^{\frac{3}{2}}} - \displaystyle\int_o^x \frac{y^2}{(1+y^2)^{\frac{3}{2}}}\,dy}{x^3} = -\frac{1}{(1+x^2)^{\frac{3}{2}}} - \frac{\displaystyle\int_o^x \frac{y^2}{(1+y^2)^{\frac{3}{2}}}\,dy}{x^3}$$

Substituting into equation (34) we obtain

$$\frac{\langle \theta_o^2 \rangle}{\langle \beta^2 \rangle} = \sqrt{\frac{2}{\pi}}\,\frac{r_o}{L}\int_{-\infty}^{+\infty} \frac{e^{-2\frac{r_o^2}{L^2}x^2}}{(1+x^2)^{\frac{3}{2}}}\,dx - 2\sqrt{\frac{2}{\pi}}\,\frac{r_o}{L}\int_{-\infty}^{\infty} e^{-2\frac{r_o^2}{L^2}x^2}\,\frac{\displaystyle\int_o^x \frac{y^2}{(1+y^2)^{\frac{3}{2}}}\,dy}{x^3}\,dx$$

The first term offers no difficulties. It has been found in the previous case (see Para. B.3).

The second integral can be written as

$$\int_{-\infty}^{\infty} e^{-2\frac{r_o^2}{L^2}x^2}\,\frac{\displaystyle\int_o^x \frac{y^2}{(1+y^2)^{\frac{3}{2}}}\,dy}{x^3}\,dx = \psi(1)$$

where

$$\psi(\lambda) = \int_{-\infty}^{\infty} e^{-2\frac{r_o^2}{L^2}x^2}\,\frac{\displaystyle\int_o^{\lambda x} \frac{y^2}{(1+y^2)^{\frac{3}{2}}}\,dy}{x^3}\,dx$$

We see by differentiation that

$$\frac{d\psi(\lambda)}{d\lambda} = \frac{1}{\lambda}\int_{-\infty}^{\infty} \frac{e^{-2\frac{r_o^2}{L^2}x^2}}{\left(\frac{1}{\lambda^2}+x^2\right)^{\frac{3}{2}}}\,dx = -\frac{2}{\lambda}\,\frac{r_o^2}{L^2}\,e^{\frac{r_o^2}{L^2\lambda^2}}\left\{K_o\left(\frac{r_o^2}{L^2\lambda^2}\right)-K_1\left(\frac{r_o^2}{L^2\lambda^2}\right)\right\}$$

It is also clear that

$$\psi(0) = 0$$

Therefore

$$\psi(1) = -\int_0^1 \frac{2}{x}\,\frac{r_o^2}{L^2}\,e^{\frac{r_o^2}{L^2x^2}}\left\{K_o\left(\frac{r_o^2}{L^2x^2}\right) - K_1\left(\frac{r_o^2}{L^2x^2}\right)\right\}dx$$

If we change the variable as follows

$$\frac{r_o^2}{L_o^2x^2} = y$$

We obtain for $\psi(1)$ finally:

$$\psi(1) = \frac{r_o^2}{L^2}\int_{\frac{r_o^2}{L^2}}^{\infty} \frac{e^xK_1(x) - e^xK_o(x)}{x}\,dx$$

As in all the previous cases it is very easy to recognize now the answer given in Section V of the Report.

TABLE I

Normalized Spatial Auto-correlation of Refractive Index	$\frac{\langle \Delta(x)\Delta(x+s)\rangle}{\langle \delta^2\rangle r_0 R}$		$\frac{\langle \beta(x)\beta(x+s)\rangle}{\langle \delta^2\rangle R/r_0}$	
	Parallel Rays	Convergent Rays	Parallel Rays	Convergent Rays
$C(r)$	$\frac{1}{r_0}\int_{-\infty}^{+\infty} C(\sqrt{r^2+s^2})\,dr$	$\frac{1}{r_0 R}\int_0^R dr_1 \int_{-\infty}^{+\infty} C\left(\sqrt{r^2+\frac{r_1^2 s^2}{R^2}}\right) dr$	$-r_0 \frac{d^2}{ds^2}\int_{-\infty}^{+\infty} C(\sqrt{r^2+s^2})\,dr$	$-\frac{r_0}{R}\frac{d^2}{ds^2}\int_0^R dr_1\int_{-\infty}^{+\infty} C\left(\sqrt{r^2+\frac{r_1^2 s^2}{2}}\right) dr$
e^{-r^2/r_0^2}	$\sqrt{\pi}\, e^{-s^2/r_0^2}$	$\frac{\pi}{2}\,\frac{\mathrm{Erf}(s/r_0)}{s/r_0}$	$2\sqrt{\pi}\,(1-2s^2/r_0^2)e^{-s^2/r_0^2}$	$2\sqrt{\pi}\left[e^{-s^2/r_0^2}(1+r_0^2/s^2)-\frac{\sqrt{\pi}}{2s^3/r_0^3}\mathrm{Erf}(s/r_0)\right]$ $\frac{2\sqrt{\pi}}{3}\left[1-\frac{9}{5}\left(\frac{s}{r_0}\right)^2\right]$ if $s \ll r_0$
$\frac{1}{1+(r/r_0)^2}$	$\frac{\pi}{(1+s^2/r_0^2)^{1/2}}$	$\frac{\pi \sinh^{-1} s/r_0}{s/r_0}$	$\frac{\pi(1-2s^2/r_0^2)}{(1+s^2/r_0^2)^{3/2}}$	$\pi\left[\frac{3+2r_0^2/s^2}{(1+s^2/r_0^2)^{3/2}}-\frac{2\sinh^{-1} s/r_0}{(s/r_0)^3}\right]$ $\frac{\pi}{3}\left[1-\frac{9}{10}\left(\frac{s}{r_0}\right)^2\right]$ if $s \ll r_0$
$\frac{1}{\left[1+\frac{r^2}{n r_0^2}\right]^n}$	$\frac{A_n}{\left[1+\frac{s^2}{n r_0^2}\right]^{n-1/2}}$ $A_n=\frac{2\pi n\sqrt{n}\,(2n)!}{(2n-1)(n!)^2\,2^{2n}}$	$\frac{A_n}{2}\,\frac{\Gamma(n-1)}{\Gamma(n-\frac{1}{2})}\sum_{\nu=1}^{n-1}\frac{\Gamma(n-\nu-\frac{1}{2})}{\Gamma(n-\nu)}\left[1+\frac{s^2}{n r_0^2}\right]^{\nu+\frac{1}{2}-n}$	$\frac{A_n(2n-1)}{n}\,\frac{1-2s^2/r_0^2}{\left[1+\frac{s^2}{n r_0^2}\right]^{n+\frac{3}{2}}}$	$\frac{A_n}{n}\,\frac{\Gamma(n-1)}{\Gamma(n-\frac{1}{2})}\sum_{\nu=1}^{n-1}\frac{\Gamma(n-\nu+\frac{1}{2})}{\Gamma(n-\nu)}\,\frac{1-2\left(\frac{s}{r_0}\right)^2\left(1-\frac{\nu}{n}\right)}{\left[1+\frac{s^2}{n r_0^2}\right]^{n+\frac{3}{2}-\nu}}$